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LETTER TO THE EDITOR

Damage spreading and multifractality in the travelling salesman problem

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Abstract. We study numerically the spreading of damage in the Euclidean travelling salesman problem. There is a critical temperature $T_c \simeq 0.20$–0.25 below which damage does not spread. We search for multifractal behaviour in the moments of the damage probability distribution. Multifractality is found in the frozen phase when the moments are monitored as a function of time. If they are evaluated with the number of cities, multifractal behaviour occurs in the chaotic phase.

The way damage spreads throughout a cooperative system is a question that arises in many fields of research. For instance, this problem has been studied in the Kauffman model [1, 2], cellular automata [3], 2D and 3D Ising models [4, 5] and spin glasses [6]. As a general rule, there is a region in the space of parameters where the damage spreads over all the system. This regime is known as the chaotic phase. On the other hand, the region where the damage does not spread defines the frozen phase. It should be remarked that this is a dynamical phase transition, which means the results might depend on the dynamics used. So, opposite results may be obtained with different dynamics; see [4, 6]. Here we study the spread of a small perturbation—the damage—in the Euclidean travelling salesman problem (TSP) in two dimensions.

Multifractality [7] has recently been observed in many different systems. The distribution of voltages in random resistor networks is a typical example of the occurrence of multifractal behaviour [8]. It has also been observed in the damage spreading for the Kauffman model [9, 10]. Each moment M_q of this distribution scales with a unique exponent which depends non-linearly on q. We also look for multifractality in the damage spreading probability distribution in the Euclidean TSP.

From a numerical study of the Euclidean TSP we find a critical temperature $T_c \simeq 0.20$–0.25 above which the damage spreads—the chaotic phase—while below it the system is in a frozen phase. We calculated several moments of the damage probability distribution and checked for multifractality in the scaling exponents. When the moments are evaluated with the number N of cities, we find clear signs of multifractality (spatial multifractality) in the chaotic phase. When the moments are monitored as a function of time, multifractality occurs in the frozen phase (time-like multifractality).

The TSP is a classic example of a complex optimisation problem—easy to formulate, yet hard to solve. It belongs to the class of NP-complete problems. The TSP is simply

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stated. A list of N cities and their distances are given. The purpose is to find a tour such that the total length l covered in travelling through all the cities and returning to the first one is minimised. There are two ways of choosing the distances between the cities: (i) the positions of the cities are random, their distances being then evaluated with the Euclidean metric; (ii) the distances themselves are random variables. This last choice is particularly suitable for analytical calculations. Simulated annealing [11] has proved to be an appropriate algorithm for finding approximated solutions: a fictitious temperature τ is introduced and a Monte Carlo simulation is performed. It has been found that many analytical tools and concepts developed in the mean-field theory of spin glasses may be useful to study optimisation problems [12–15]. In fact, there are analogies between spin glasses and the TSP [14, 16].

We simulated the TSP with Euclidean distances using the Metropolis algorithm. The length l plays the role of an energy. In order to have well defined thermodynamical quantities [12], the temperature τ and the length l should be normalised as follows: $l = L\sqrt{N}$ and $\tau = T/\sqrt{N}$. The basic Monte Carlo updating strategy is a two-bond or 2-opt replacement [17]. In this operation two cities chosen at random exchange their positions in the tour. The cities between them are traversed in the opposite direction. This procedure has the effect of 'flipping' the subsection with respect to the entire tour. This change is accepted without further conditions if $\Delta l \leqslant 0$ and with a probability $\exp\{-\Delta l/\tau\}$ if $\Delta l > 0$. The simulation proceeds until the system is in equilibrium at a given temperature T. Then, at time $t = 0$ we make two replicas, B and C, of the original system A. The initial tours in replicas B and C are obtained by introducing a 2-opt transformation in B and C and another one in C. An order parameter can be defined in the following way. If Q_{AB} is the overlap between replicas A and B—i.e. the fraction of common bonds—then Q is given by [4]

$$Q(t) = Q_{AB}(t) + Q_{BC}(t) - Q_{AC}(t).$$

Obviously, $Q(0) = 1$. If the damage spreads $Q(t)$ will decrease with time. Then, in the chaotic phase, $\lim_{t\to\infty} Q(t) < 1$. In the frozen phase $\lim_{t\to\infty} Q(t) = 1$. It is convenient to work with the related order parameter

$$\psi = -\log Q(\infty)$$

which makes the evaluation of the critical temperature easier. $\psi(T)$ is shown in figure 1 for $N = 49$. Clearly, there is a sudden change in its behaviour at $T_c \simeq 0.20$–0.25. Below T_c the damage does not spread: the system is in a frozen phase. Above T_c damage spreads: it is in a chaotic phase. The behaviour of T_c with N was analysed but no appreciable dependence was found in the range between $N = 49$ and $N = 196$. It should be remarked that close to T_c the dynamics exhibits strong fluctuations, which makes the estimation of the critical temperature difficult.

Next we analyse the appearance of multifractality at the onset of the chaotic transition. Let us consider the probability that a site has been damaged at time t

$$p_i(t) = n_i(t)\left(\sum_j n_j(t)\right)^{-1}$$

where $n_i(t)$ is the number of times site i has been damaged up to time t and the sum in the denominator runs over all damaged sites.

These probabilities are easily measured; once they are known their distribution is given by

$$P(p) = \sum_i \delta(p - p_i)$$

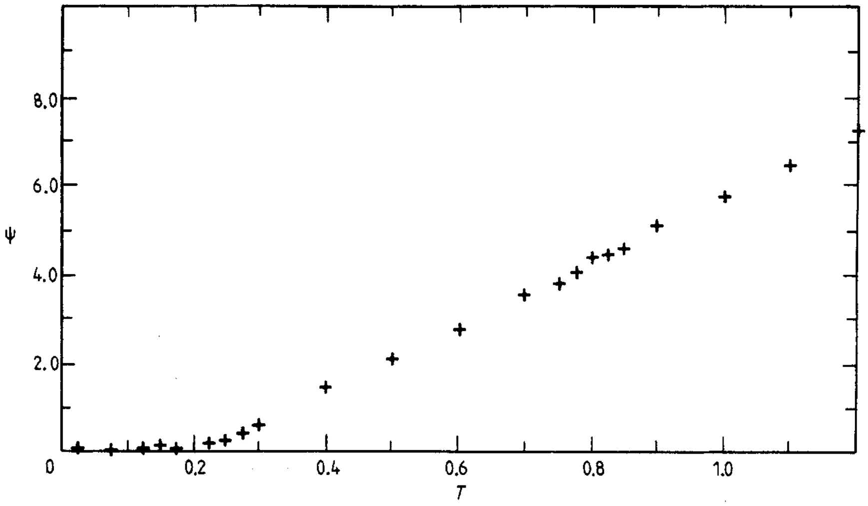

Figure 1. Order parameter ψ as a function of T for $N=49$. There is a clear change of behaviour at $T \simeq 0.20$–0.25.

and its moments M_q are

$$M_q = \sum_i p_i^q.$$

It is expected that as a function of time $M_q \sim t^{\phi(q)}$. In the chaotic phase, damage spreads easily with a constant propagation velocity. Most sites will damage in accordance with $n_i \propto t^{\alpha}$. If the total number of damaged sites M_0 behaves as a fractal, then $M_0 \sim t^{d_t}$ with d_t the fractal dimension. In this case, $\Sigma\, n_i \sim t^{d_t+1}$ and $M_q \sim t^{d_t(1-q)}$. If these assumptions are right, then $\phi(q) = d_t(1-q)$. In figure 2, $\phi(q)$ is shown as a function of q for $T=1.00$ and $N=49$. It is a straight line; the cluster of damaged sites grows as t^{d_t} with $d_t \simeq 1$. At the onset of the phase transition and inside the frozen

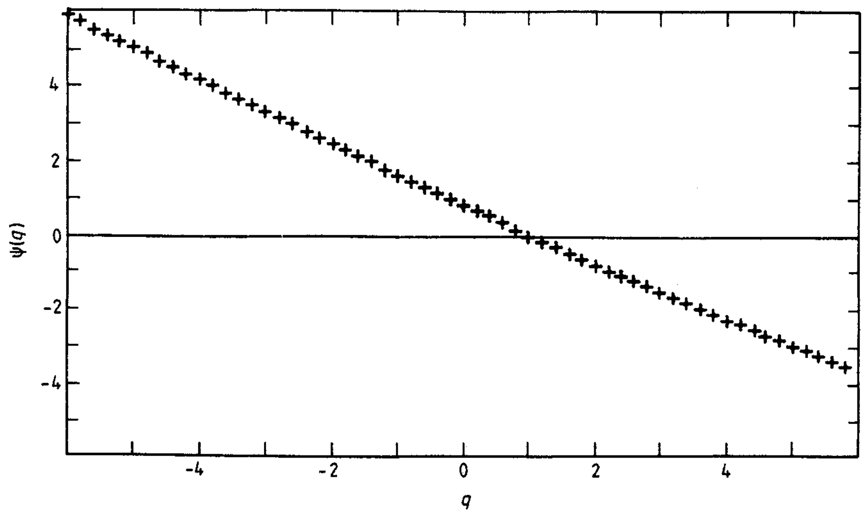

Figure 2. Time-scaling exponents $\phi(q)$ as a function of q; $T=1.00$ and $N=49$.

phase the linear behaviour disappears. In figure 3 we show our results for $T = 0.25$. We also checked the existence of multifractality for $T = 0.15$; the corresponding curve is indistinguishable from figure 3. This implies that the set of all damaged sites does not satisfy the simple relation $n_i \propto t^{\alpha}$; instead it can be decomposed into many fractal subsets, each one made of sites characterised by a fractal exponent α_i. Thus, in the frozen phase there are clear signs of multifractality in time.

We also checked multifractality in terms of the number N of cities. We computed the moments M_q at time t_0 such that all the bonds have been damaged at least once. These measurements are taken at $T > T_c$. It is beyond our computational capacity to compute $M_q(t_0)$ at $T \simeq T_c$. For large N, one expects: $M_q \sim N^{\theta(q)}$. Again, the number of damaged bonds scales as $M_0 \sim N^{d_e}$. If the cluster of damaged bonds behaves as a fractal, then $M_q \sim N^{d_e(1-q)}$. We measured M_q for several values of N ranging from 16 to 100. $\theta(q)$ is shown in figures 4(a) and 4(b) as a function of q for $T = 1.00$ and $T = 0.35$. It is not a straight line. This implies the existence of multifractality. Let us remark that the curvature for the case $T = 1.0$ is less than for that at $T = 0.35$; however, it is not a straight line as in the case of scaling with t (see figure 2).

Our results should be compared with those found in other systems. In the Kauffman model, there is multifractality in time at the critical point but it is absent in the chaotic phase [9]. As we showed, the same conclusion is valid for the TSP. There is no spatial multifractality in the Kauffman model [10] but it is certainly present in the chaotic phase of the TSP. Very recently, spatial multifractality was found at the onset of the transition of the 3D Edwards–Anderson spin glass [18]. On the other hand, it was not found either in the 2D version of the same model or in the 3D Ising ferromagnet [18]. Our work shows that from the point of view of multifractality the TSP is similar to the 3D Ising spin glass.

In summary, we have studied numerically the damage spreading in the two-dimensional Euclidean TSP. We have found a critical temperature $T_c \simeq 0.20$–0.25 below which there is no spreading of damage. We have determined the moments of the bond damage probability distribution. The scaling properties of M_q with t have been tested. We found multifractality for $T < T_c$. For $T > T_c$, the moments show a simple fractal

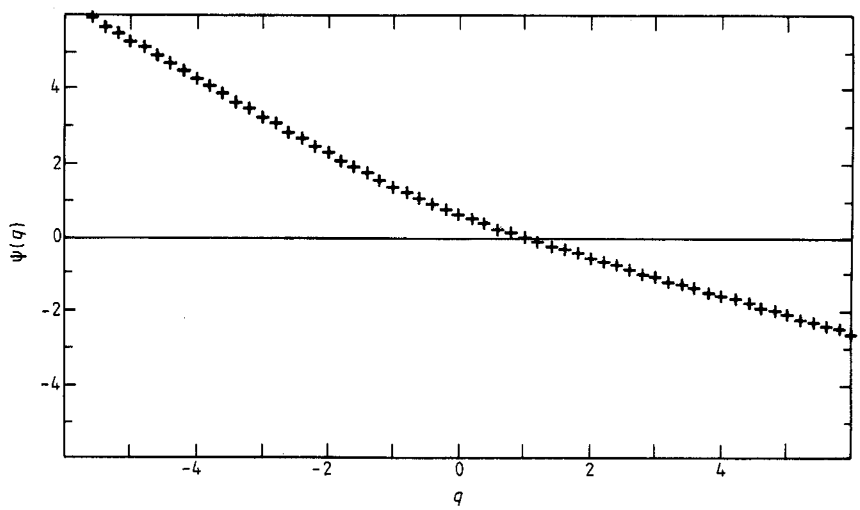

Figure 3. Exponents $\phi(q)$ as a function of q for $T = 0.25$ and $N = 49$.

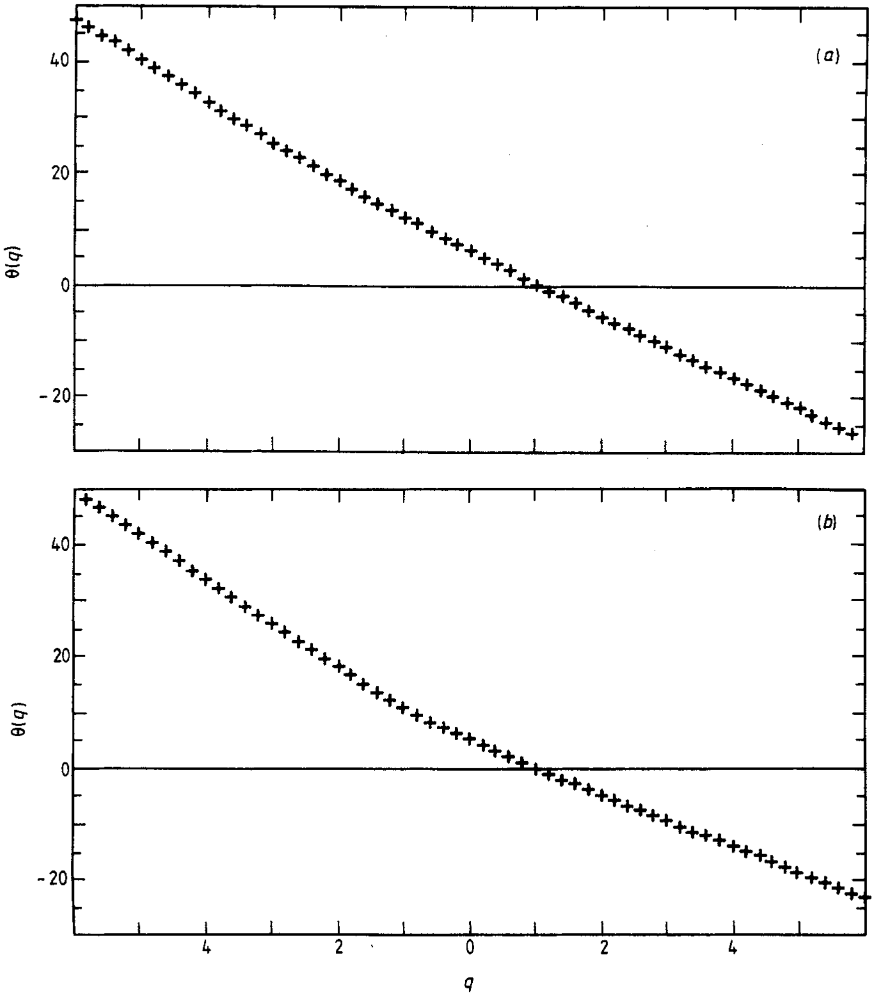

Figure 4. Spatial scaling exponents $\theta(q)$ as a function of q for (a) $T = 1.00$, (b) $T = 0.35$.

behaviour with $d_t \simeq 1$. We have also studied the scaling of the M_q with N, finding evidence of multifractality in the chaotic phase.

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